# János Bolyai, the founder of the Non-Euclidean Geometry

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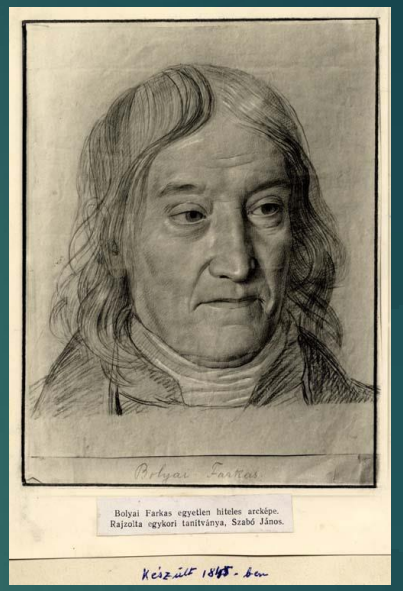
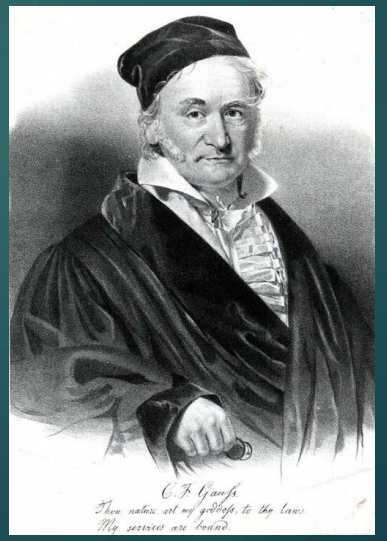
# His father, Farkas Bolyai was a mathematician

- Farkas Bolyai had the chance to go as an accompany of the son of Baron Kemény to Jena, where Bolyai for the first time began to study mathematics systematically. After six months in Jena Bolyai and Kemény went to Göttingen. There he was taught by Kästner and became a life long friend of Gauss, a fellow student at Göttingen. This was the time when one could say that Bolyai really became a mathematician.
- He began to think about Euclid's geometrical axioms and in particular the independence of the Fifth Postulate. He discussed these issues with Gauss and his later writing show how important he considered their friendship to be for his mathematical development.
- By the autumn of 1798 Bolyai and Kemény had completed their studies, the latter returned to Hungary, but Bolyai was left penniless in Göttingen. He spent a year there relying on charity and borrowed money for food to survive. It was a year of great hardship, yet one where he continued to develop mathematically surrounded by other talented mathematicians. After a year a friend sent him enough money and he returned on foot as well in July 1799.



# János Bolyai was born in Kolozsvár, 1802

- Back on the family estate at Bolya, Farkas undertook research in mathematics. He went to Kolozsvár where he became a tutor. There he met Zsuzsanna Benkő and they married in 1801. In Zsuzsanna's parents home on 15 December 1802 their son János Bolyai was born. His native house has been marked with a memorial plaque since 1903.



# Farkas Bolyai became professor in the Calvinist College in Marosvásárhely for about 50 years

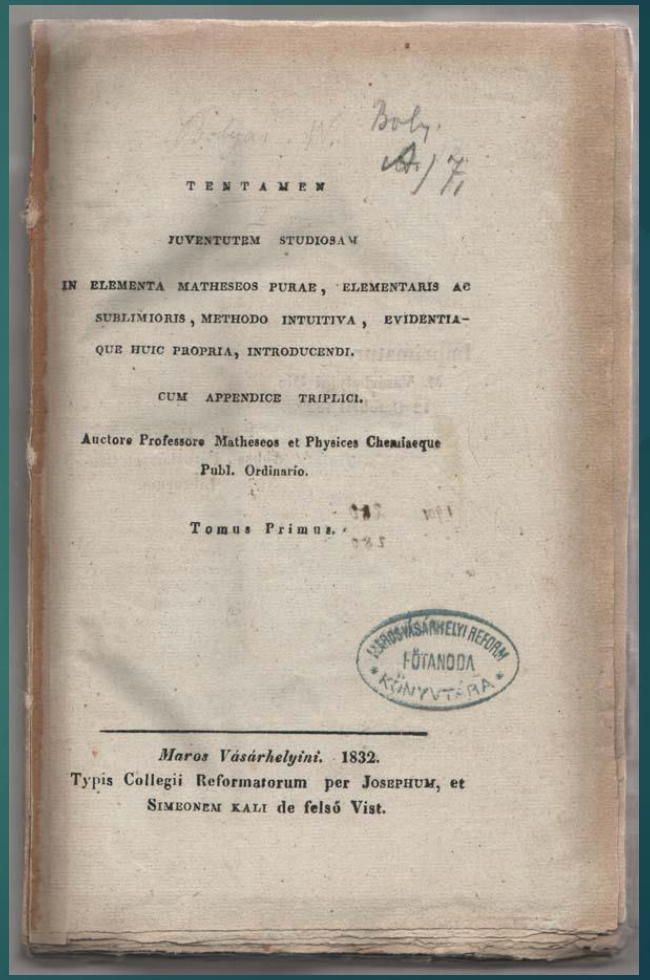
- Farkas Bolyai was offered a job at the Calvinist College in Marosvásárhely, and he moved thare with his family. Bolyai taught mathematics, physics and chemistry at Marosvásárhely for the rest of his life.
- Life was not easy for Bolyai in Marosvásárhely. He was paid very little for his teaching at the College and had to take on extra work to bring in extra money.
- He wrote and published dramas, he ran the College pub, and he designed tiles and cast iron stoves which were produced commercially.

A MAROSVÁSÁRHELYI EV. REF. KOLLEGIUM 1777—1803-BAN ÉPITETT RÉSZE.

BOLYAI FARKAS LAKÓHÁZA MAROSVÁSÁRHELYT.

# Farkas Bolyai, the mathematician

- His main work is the *Tentamen* whose first volume was published in 1832, the second in 1833. His mathematical oeuvre focused on the attempt to demonstrate the 5th postulate of Euclid. It was later his son János to point out that the 5th postulate could not be demonstrated from the rest of the axioms of Euclid.
- The research of Farkas Bolyai is nevertheless very important because of his recastings of the postulate.
- The problem which had perplexed Bolyai most in his study of mathematics had been the independence of Euclid's Fifth postulate. In 1804 he believed that he had a proof that it could be deduced from the other axioms, but he sent his proof to Gauss who discovered the error. Eventually he gave up his attempts to prove its independence and tried instead to find an equivalent version which was more easily accepted by common sense. The *Tentamen* contains eight axioms equivalent to Euclid's Fifth Postulate such as:
- *- Three points which do not lie on the same straight line must lie on a circle.*
- *- No sphere may differ from any other sphere in any property except its size and location.*
- He also made remarkable achievements in the research of the theory of sets, the convergence of infinite series and the finite equivalences of surfaces.


TENTAMEN

JUVENTUTEM STUDIOSAM

IN ELEMENTA MATHESEOS PURAE, ELEMENTARIS AC SUBLIMIORIS, METHODO INTUITIVA, EVIDENTIAQUE HUIC PROPRIA, INTRODUCENDI.

CUM APPENDICE TRIPLICI.

Auctore Professore Matheseos et Physices Chemiaeque Publ. Ordinario.

Tomus Primus.

*Maros Vásárhelyini.* 1832.
Typis Collegii Reformatorum per JOSEPHUM, et SIMEONEM KALI de felső Vist.

# The young Bolyai, between 1806-1814

- **Between 1806–1814**: János at the age of five already knew several stars, and could draw some constellations by memory. At the age of six he started to read almost by himself. One year later he began to learn to play on violin, and at the same time he started to learn German and Latin.
- He began his regular studies at the age of nine, but his father at this time did not send him to the College yet, but the senior students came to their home to teach him. In mathematics he was instructed by his father.
- **In 1814** János became an ordinary student of the Calvinist College of Marosvásárhely. He was admitted to the 4th class.



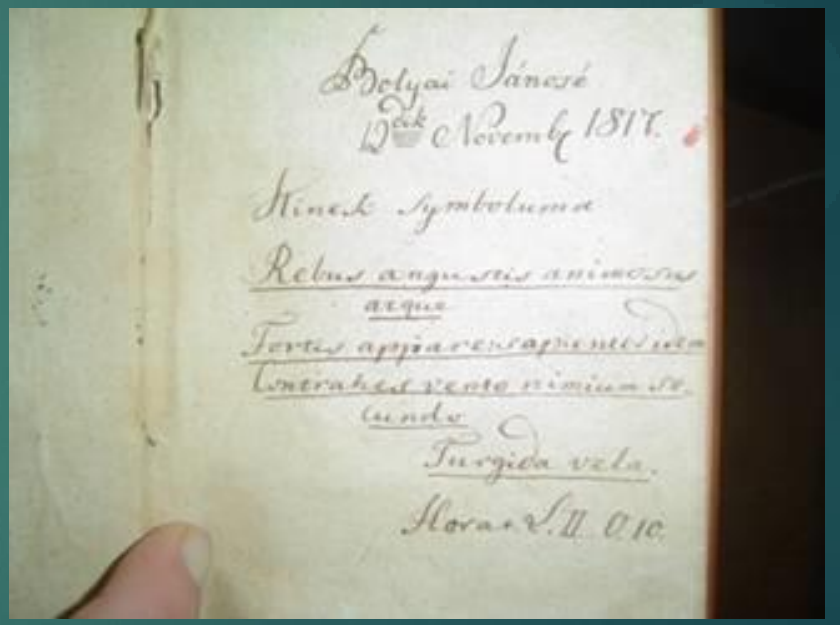
A memento of János Bolyai from the collection of Dániel Vajda. Marosvásárhely, August 3, 1814. The Latin text in English: "Be temperate in happiness and not lavish in crying either. / Our life waves between joy and woe." (From the Fables by Phaedrus.) Autograph, 1 fol. LHAS Department of Manuscripts and Old Books K 25/10. 5.

# The Letter of Farkas Bolyai to Gauss

- **On April 10, 1816** Farkas Bolyai asked for the approval of his university friend Carl Friedrich Gauss to send János to him at the university of Göttingen. In his opinion at the universities of Pest and Vienna the level of the mathematical courses was not high enough to seriously evolve the talents of János.
- In his letter he characterized his son like this: „I started with Euclid, then he got to know Euler. By now he completely knows the first two volumes of Vega (which I use in my courses), and he is also well versed in the third and fourth volumes. He loves differential and integral calculus, and does them with an extraordinary ability and ease. […] He perceives much and quickly, and he often has highly gifted intellectual glances."
- Gauss did not reply to this letter.
- **At the end of 1816** Farkas thought to send his son to a military officer's career, and János also agreed with the idea.
- **On July 30, 1817** János passed his final exam in the Calvinist College of Marosvásárhely as the first of his year.

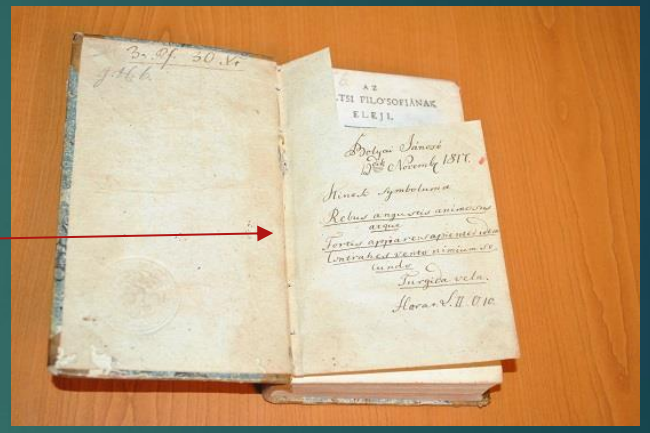
# The ownership signature of the 15 years old János Bolyai

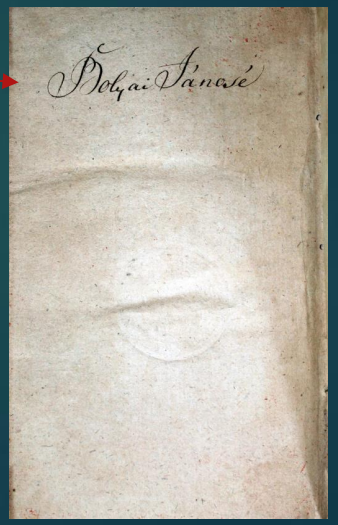
- It seems to me that some unexpected research results may appear in connection with Bolyai all the time.
- Almost two years ago (8th Nov. 2013.) I was invited for a traditional Bolyai-event, a festive dinner, organised by the University of Cluj-Napoca. As an ex-student of them,– I was walking around the building of the university in Farkas Street.
- Coming from the building of the old library behind the university in a tiny street I noticed a small second-hand bookshop, and the same atmosphere with the shelves full of books and the small, narrow corridor reminded me of being among a late friend of mine, the last polyhistor of Transilvania, Ádám Müller's -books again.
- The saleswoman did speak only Romanian and English, but she is ready to show the few books about Mathematics. "Unfortunately, we have nothing in Hungarian" she says. I asked her to give me the only book that seemed exciting, which turned out to be not so intertesting: a French author, German publication from the beginning of the century.
- There is something you might find interesting, – she says – realizing I fancy old books. A Bolyai ex libris can be found in an old book. I would really like to see it, if possible- I replied.



# If you know his handwriting, you can see it is original – she said

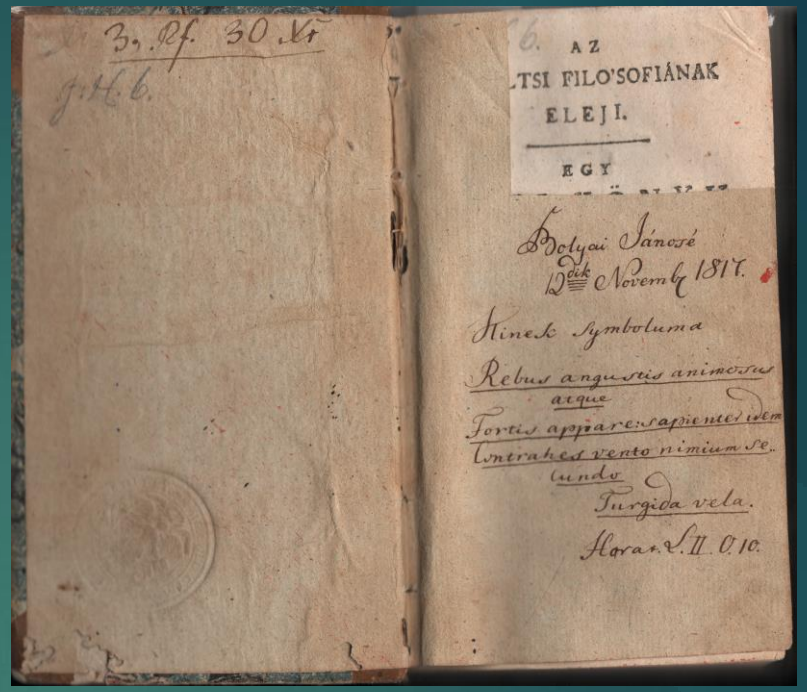
- They had a small cabinet with a lock for the more valuable volumes. She took the book from there, and being a perfect merchant she was praising it: – „Look, here is a note, written by Bolyai. If you know his handwriting, you can see it is original. There is something written in Latin as well, obviously by him. What is more important the fact that the owner is really Bolyai himself, is proved by the remark on the last page. On the back of the very last page there is the same script with a bit smaller letters by János Bolyai.”
- I was just standing there, staring at it and I tried to believe my eyes.
- Did you know that Bolyai was only 15 years old at that time? – I asked her. There is a commemorative plaque on his house of birth in the street round the next corner.
- She had no idea and maybe she didn't care at all.

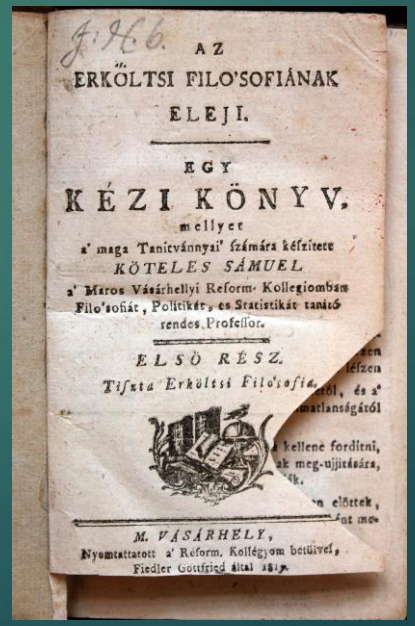


# What to do?

- I had a suspicion: we all heard that some volumes had mysteriously disappered from the Teleki-Bolyai Library in Targu Mures. My suspicion was not reduced by the fact that on the first page, above the note a small, almost square-shaped, 4–5-cm wide part was cut, and on the bottom of the second page an approximately 1-cm wide stripe was cut, too.
- On the first pages there was the mark of a relief printing seal, but unfortunately its label was impossible to read. I knew I had to be careful so I asked permission to take some photos, – in my opinion a library is entitled to buy it, – but the permission was refused. Her boss from the back room allowed me to make notes about its main characteristic features only: the date of the script, the year of its publication,  and the fact that the book was printed with the letters of the Reformed High School in Targu Mures - as later on the Tentamen, and Appendix

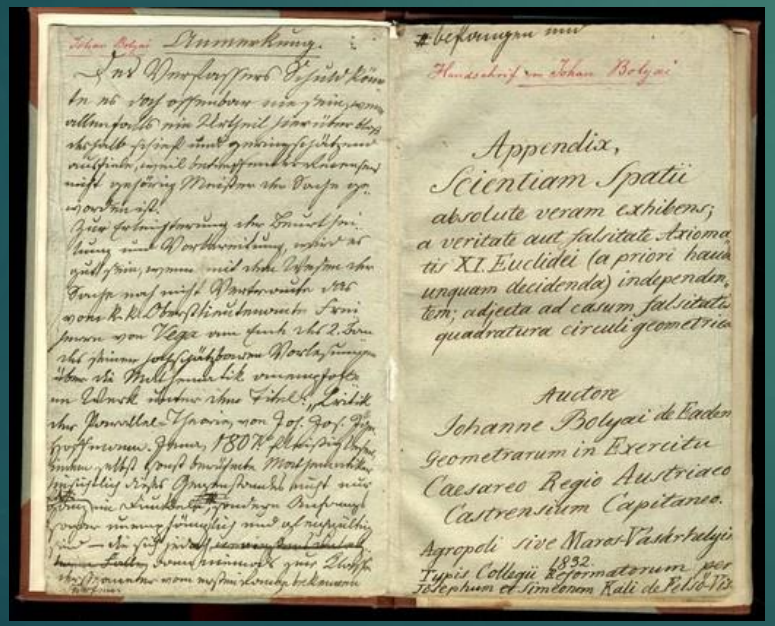



# My doubts

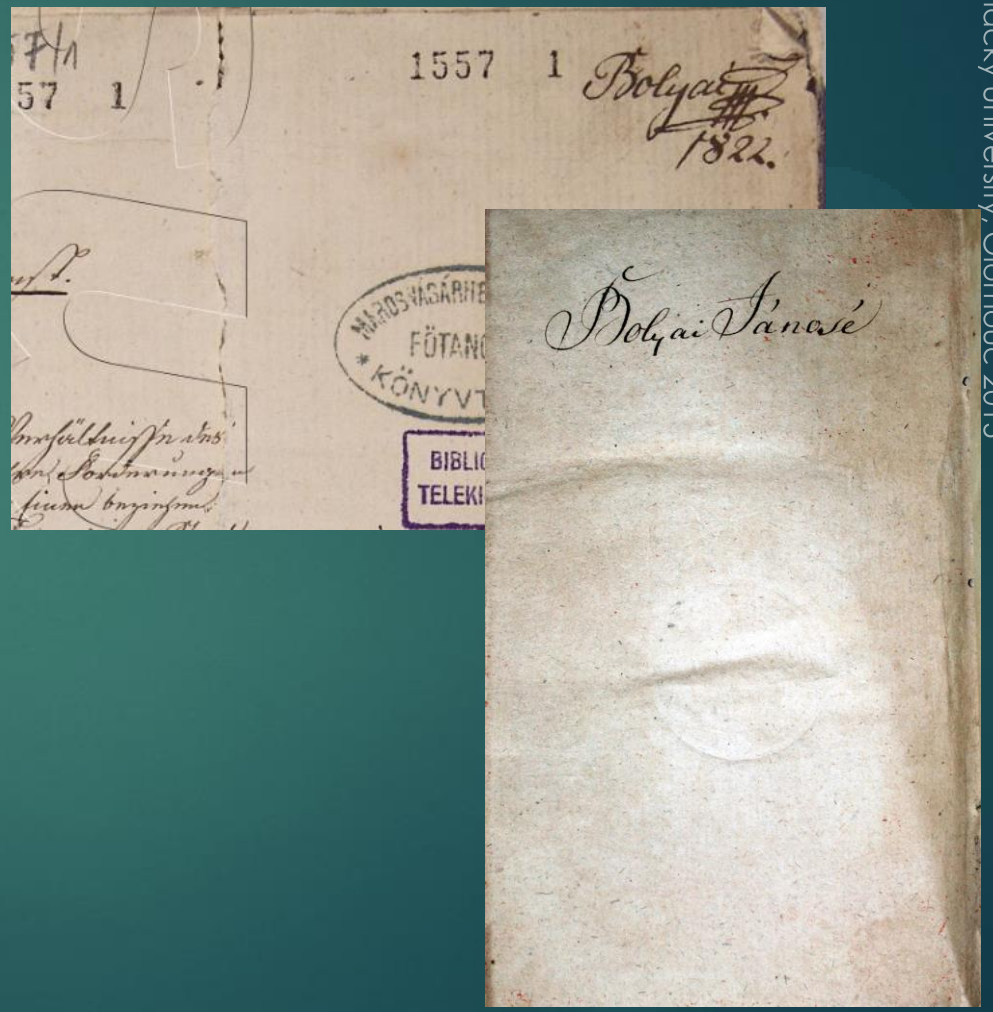
- However I was sure that the manuscript is original because luckily I had studied Bolyai's handwriting a lot. I am one of the few, who is able to read Bolyai's more and more complex handwriting
- I decided to share my discovery with a group of my friends and acquaintances arriving for the Bolyai dinner party. Some of them had the same idea as me: maybe I have found one of the volumes having disappeared from the Library. It is obvious from the missing parts cut out. In this case (the book is original) they felt the same enthusiasm. When I asked them who is willing to search for the further details, coping with the possible failure or hardship, only Csaba Szabó, journalist and publisher was keen on my idea.
- He got my notes with all the identification details. I asked him only to provide me with all the information about what is happening later and in case of writing an article to mention my name as the finder of the book and the script.

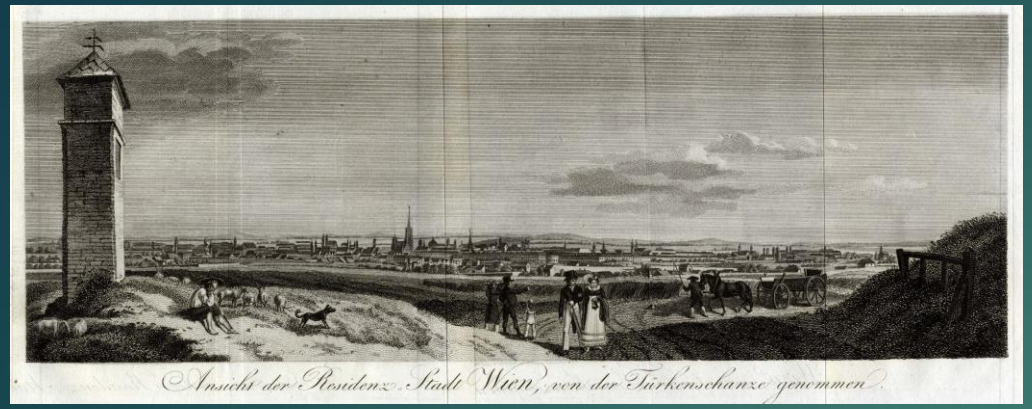
- Here is a rare species of the Appendix, one of the pages from the original manuscript, where Bolyai in consciously writing beautifully. I had this page and the title pages of "Üdvtan" in my mind in the second-hand shop when I believed my eyes:
- The titlepage of the Appendix*, on the left hand side: Anmerkung, Bolyai's critical notes, on the right hand side: János Bolyai's own plan for the titlepage, the red notes are Ferenc Schmidt's remarks,

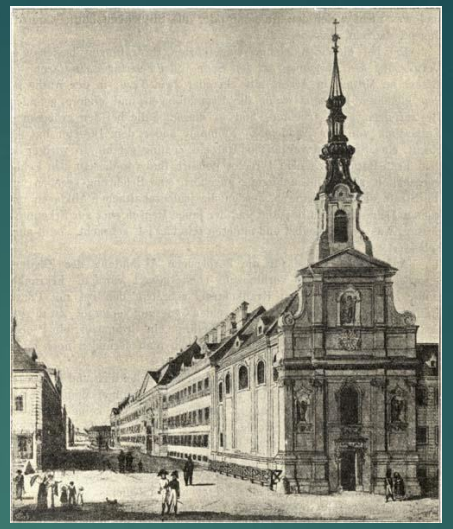


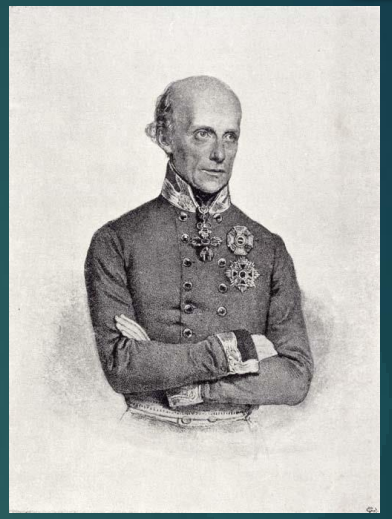

* 545.091, Library of the Hungarian Academy of Sciences.

## Other interesting pages with Bolyai's signature:





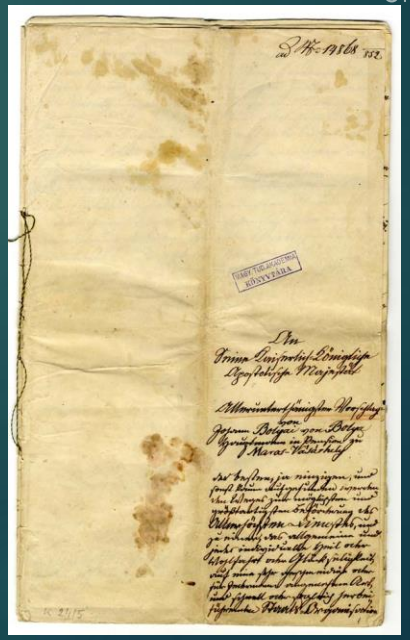




**On August 24, 1818** he passed a successful exam of admission at the Imperial and Royal Academy of Engineering, founded in 1717. As the sixth one on the order of results, he was directly passed to the 4th class, the highest level one could reach by exam. The academic studies embraced eight years, and thus he only had to count with four years of schooling.

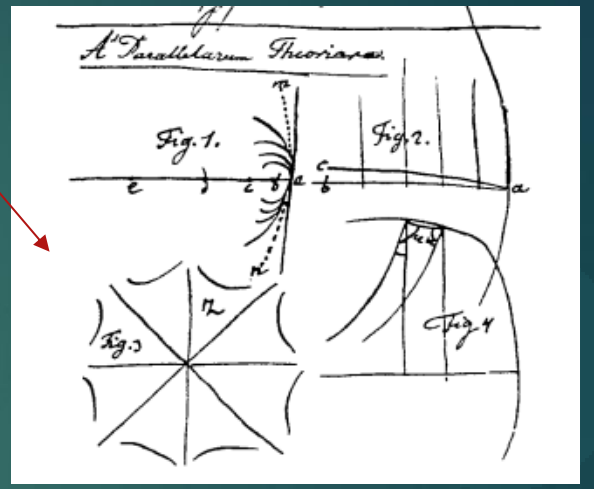
His studies were supported among others by Count Miklós Kemény, the main guardian of the College of Marosvásárhely, and Count Ádám Kendeffy

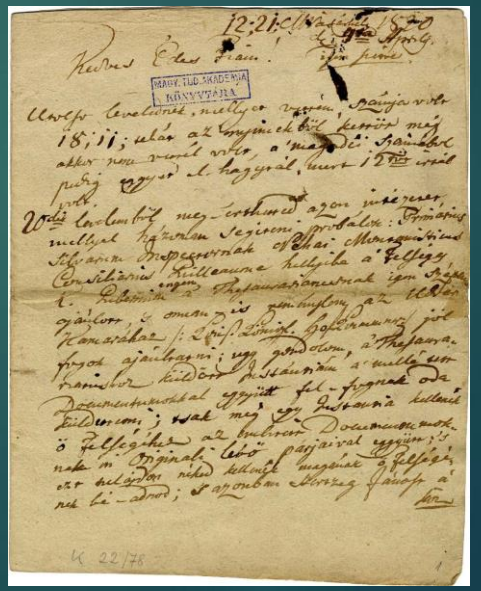
**Before April 1820,** probably in the academic year of 1819–1820 Archduke Johann – the younger brother of Emperor Franz I, and the chief director of the Academy – on an official visit took notice of the mathematical talent of the younger Bolyai, and as the supervisor of the engineering corps he did not forget about him later.



# 1820, Vienna, he wrote his first remarks on the Theory of Parallels – A Parallelarum Theoriára

- **In the spring of 1820** he informed his father about his experiences concerning parallels.
- Farkas in a long letter warned his son against this: "*…don't go any step further, or else you're a lost person…Do not try the parallels in that way: I know that way all along. I have measured that bottomless night, and all the light and all the joy of my life went out there.*"
- **On September 6, 1822** he completed the last semester of the Academy with excellent certificate, as the second best student of his year. (The order of excellence of the students was established by the professors and the students in common, and although the professors considered János as the best, the vote of the students was decisive.) The two best students, thus also János were kept at the Academy for one more year as "Ingenieur-Corps Cadets" for supplementary studies, especially important for military architects. Their subjects were: the second part of fortification, architecture, plans of fortification and architecture, German writing and French.



# 1822, Vienna, 1823, Temesvár

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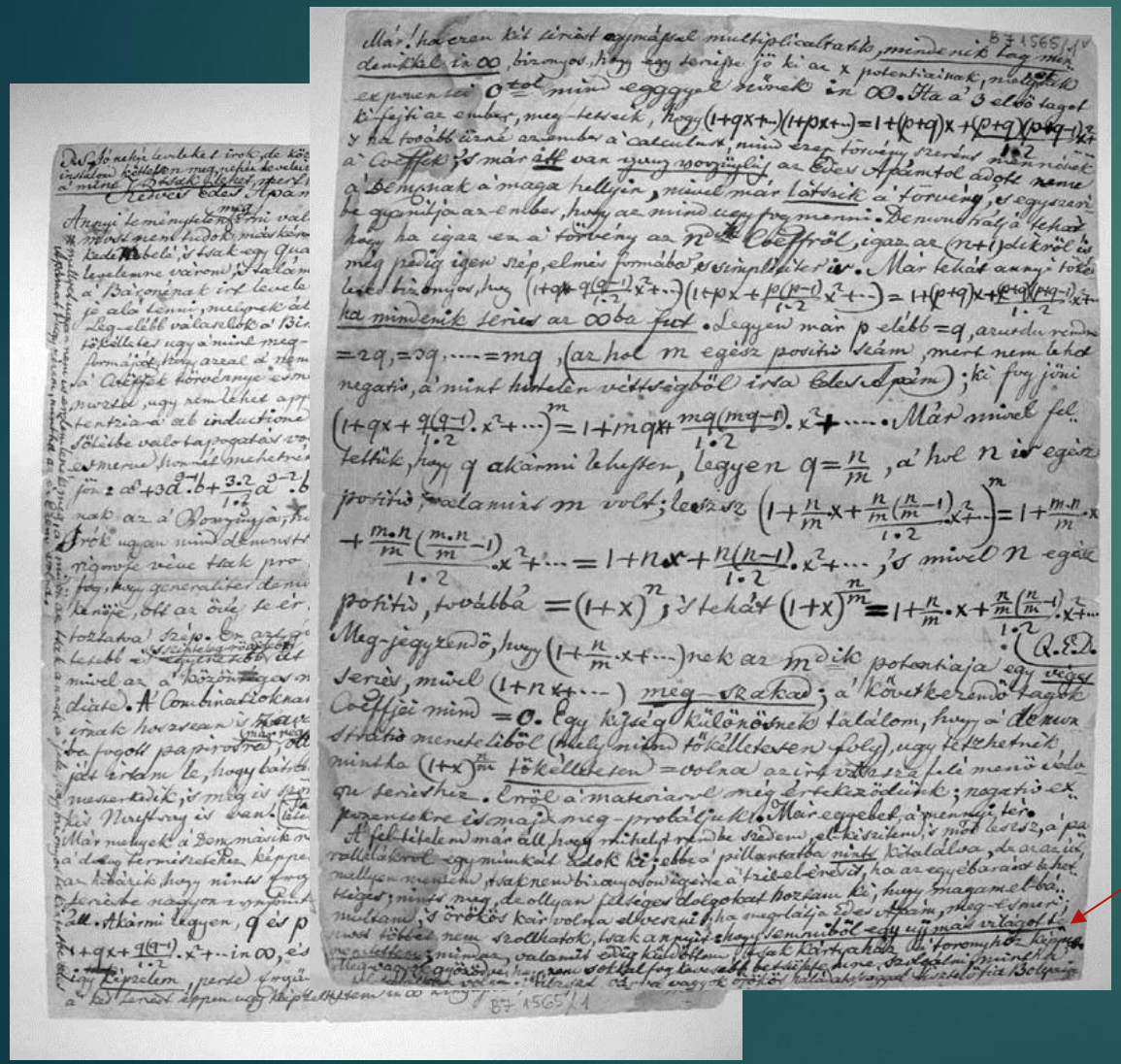
- **On September 1, 1823,** after the year of cadetship János was appointed sub-lieutenant and commanded to the directorship of fortification in Temesvár (Timişoara).

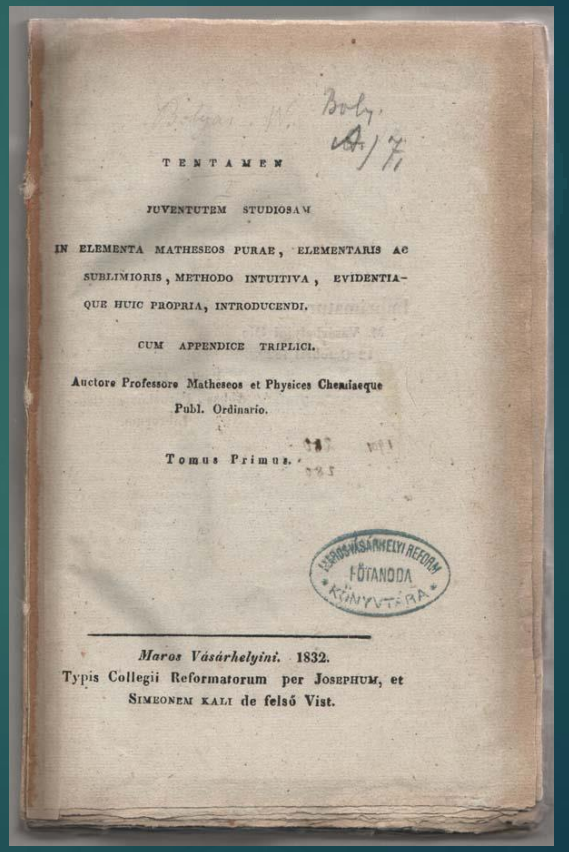
# Bolyai's Temesvár Letter to his father, 23 November, 1823



I have got already the conditions, and once I put them in order, and finish them, I will
publish a paper about the parallels; right at this moment is not finished, but the way I followed, is almost sure promising to attain the goal, if it were ever possible; it is not finished but I have found beautiful things, that surprised even me, and it would be a pity to lose them; my Dearest Father will see and know; I cannot say more, only that from nothing I have created a new different world; Everything I sent you before is like a house of cards if compared to a tower. I am convinced, that it will be in my honour, not less than if I would discover ... Waiting for your answer, yours for ever indebted son Bolyai.

# 1826, Arad, Eckwehr


- **From April 10, 1826** he began his service as sub-lieutenant at the directorship of fortification of Arad.
- His superior was Johann Wolter von Eckwehr, his former professor of mathematics at the Academy of MilitaryEngineering, with whom he had been in correspondence.
- In 1826 János gave over to him his manuscript treatise in which he summarized his researches on non-Euclidean geometry (this manuscript has been lost).
- **In December 1827 – January 1828** his health turned worse. "Lieutenant János Bolyai asked six weeks of leave to his father in Marosvásárhely because of his convalescence after illness…"
- **At the end of January 1828** János was already in Marosvásárhely, and he saw the manuals of mathematics of his father prepared for publication: *The principles of mathematics* (Marosvásárhely, 1830) and *Tentamen*. (The manuscript of this latter had been prepared by as soon as 1829.)
- Probably also János presented to his father his German treatise summarizing his results in geometry, and their exchange of views may have inspired János to deepen his research concerning the space of absolute geometry.
- **Around April 25, 1828** he went back to the place of his service in Arad.



TENTAMEN

JUVENTUTEM STUDIOSAM

IN ELEMENTA MATHESEOS PURAE, ELEMENTARIS AC SUBLIMIORIS, METHODO INTUITIVA, EVIDENTIAQUE HUIC PROPRIA, INTRODUCENDI.

CUM APPENDICE TRIPLICI.

Auctore Professore Matheseos et Physices Chemiaeque Publ. Ordinario.

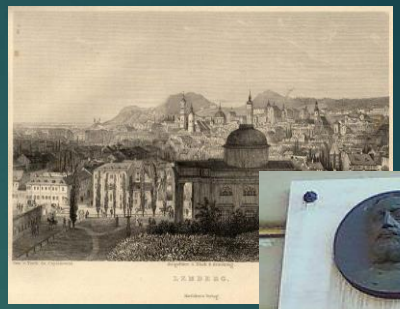
Tomus Primus.

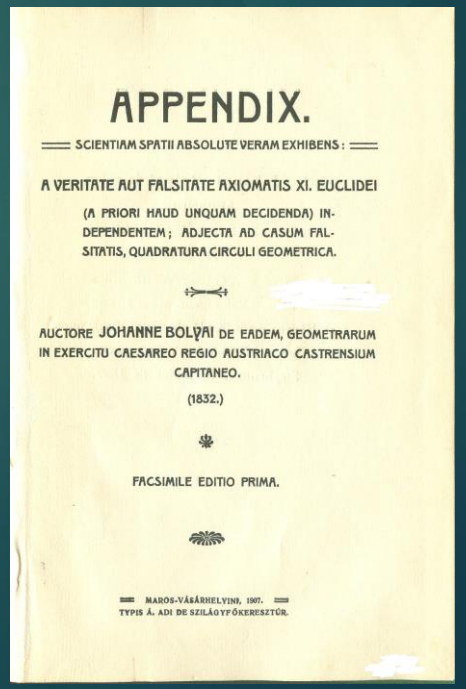


*Maros Vásárhelyini.* 1832.
Typis Collegii Reformatorum per JOSEPHUM, et SIMEONEM KALI de felső Vist.

# Appendix was published as a separate annex to Tentamen

- **From December 8, 1830** he was appointed the chief engineer of the directorship of fortification of Lemberg (Lvóv), but he occupied his post of service only six months later.
- **From the beginning of December 1830 until May 1831** he spent his holiday in Marosvásárhely, which was decisive in the preparation of his *Appendix,* as his father insisted on his putting it on paper.
- Some weeks after his departure from Marosvásárhely, on July 13, 1831 his father wrote in a letter to Lajos Jakab on the space theory of János: "It is an original great work that will be appreciated everywhere; no similar work on mathematics has been written by any Hungarian author…"
- During this time they decided that the work of János will be included in the first volume of the *Tentamen* as an *Appendix.*
- **In the middle of July 1831** Farkas Bolyai did not wait until the *Appendix* can be published together with the *Tentamen,* but published the treatise of János in a separate small volume.





APPENDIX.

SCIENTIAM SPATII ABSOLUTE VERAM EXHIBENS:

A VERITATE AUT FALSITATE AXIOMATIS XI. EUCLIDEI (A PRIORI HAUD UNQUAM DECIDENDA) INDEPENDENTEM; ADJECTA AD CASUM FALSITATIS, QUADRATURA CIRCULI GEOMETRICA.

AUCTORE JOHANNE BOLYAI DE EADEM, GEOMETRARUM IN EXERCITU CAESAREO REGIO AUSTRIACO CASTRENSIUM CAPITANEO.

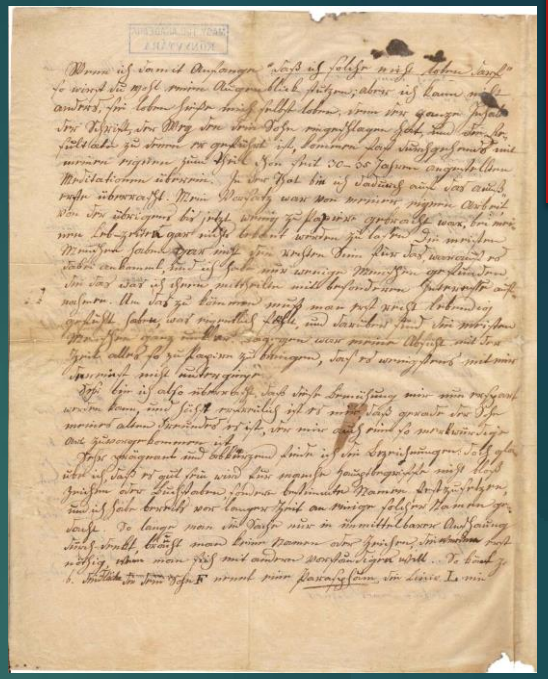
(1832.)

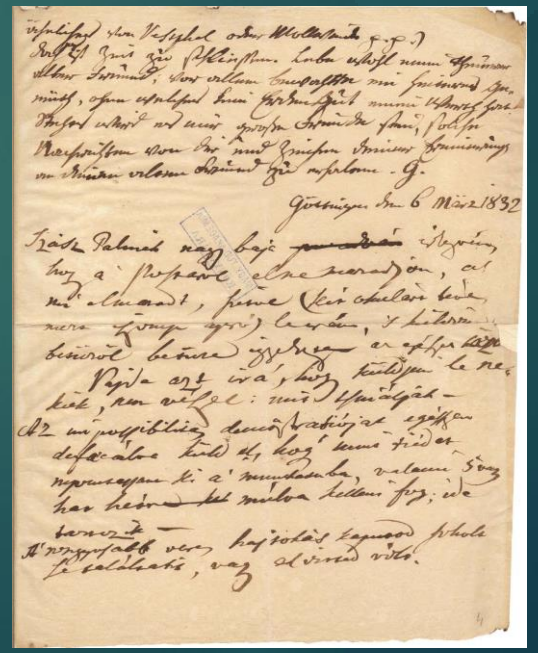
FACSIMILE EDITIO PRIMA.

MAROS-VÁSÁRHELYINI, 1897.
TYPIS Á. ADI DE SZILÁGYFŐKERESZTÚR.

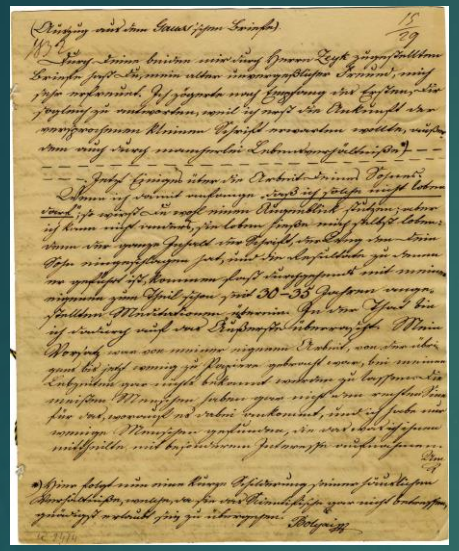
# The Appendix and Gauss

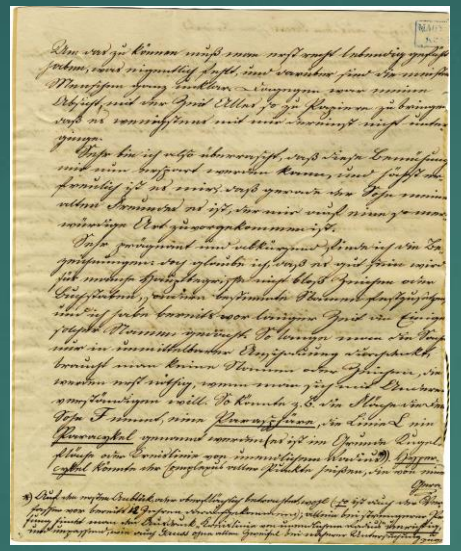
- His father immediately sent a copy of the *Appendix* by post to Gauss, asking for his opinion, but it was lost on the way.
- **On January 16, 1832** Farkas sent the treatise of his son to Gauss again.
- **On March 6, 1832** Gauss replied that the way János had been following and the results achieved by him coincided with his own research done in the previous 30-35 years and that he himself wanted to compose a similar work but he did not intend to publish it in his own life.
- Farkas had made a copy of this letter for his son who received it around April 6, 1832 in Lemberg. This letter unsettled János, and the offence caused by it never passed
- Nevertheless, Gauss acknowledged in a letter of February 14, 1832 written to Christian Ludwig Gerling that his ideas had not been as mature as those of the young Bolyai, and he also added: "I consider this young geometer, Bolyai, a first-class genie."

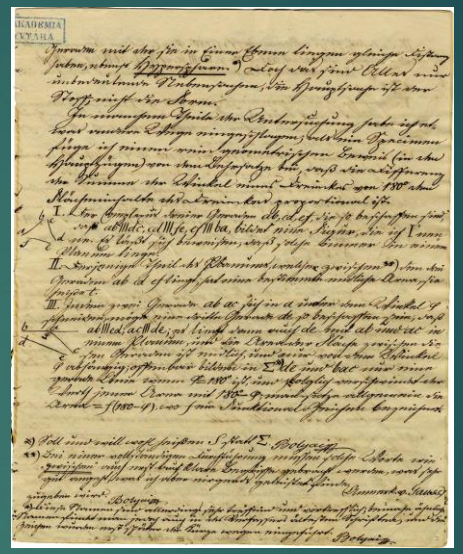


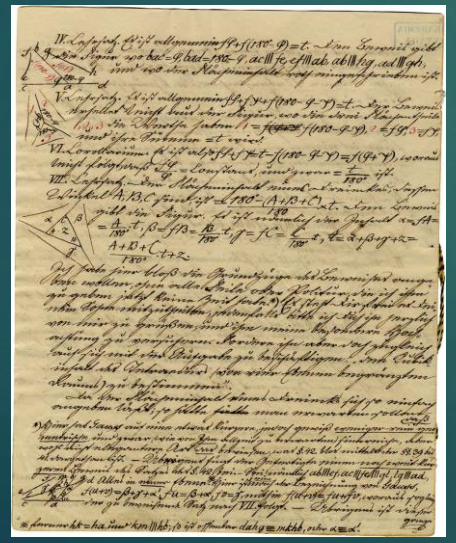


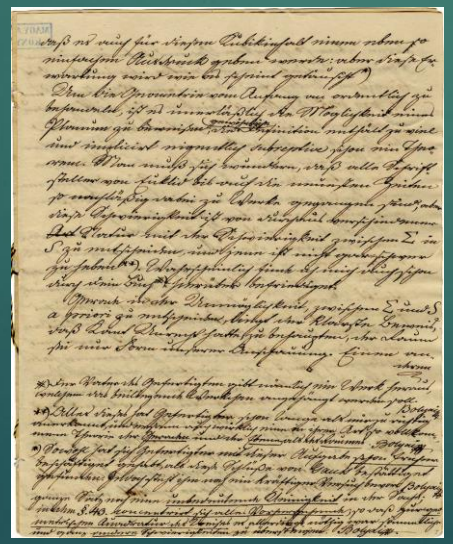
# The copy of the letter of Gauss

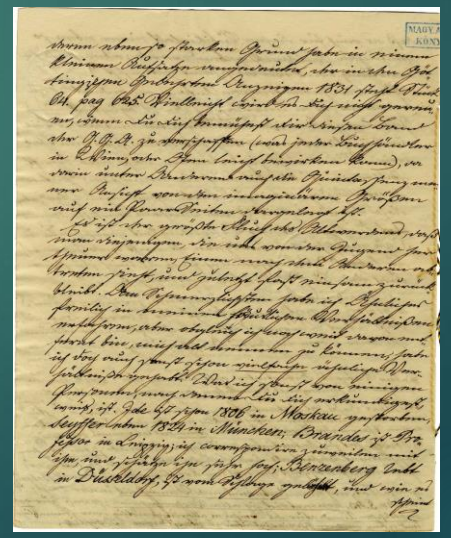






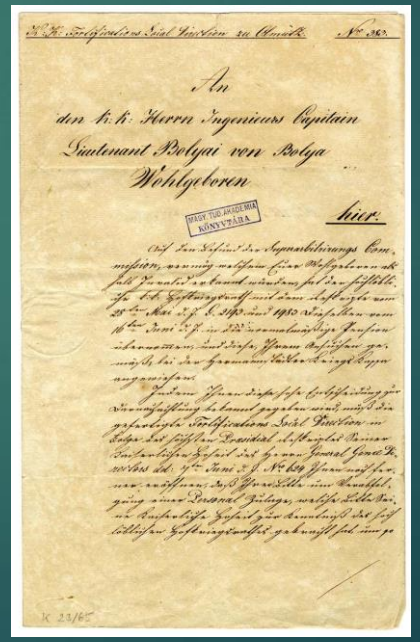


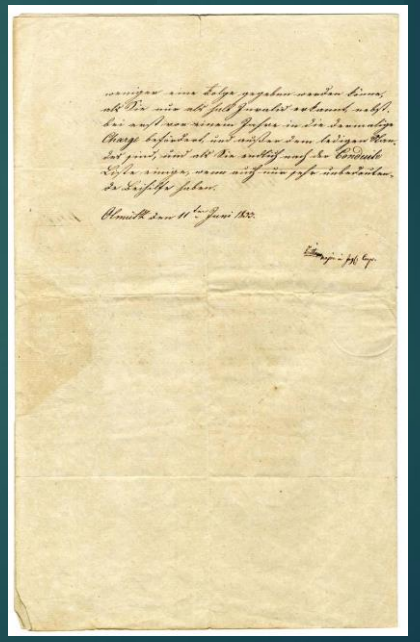




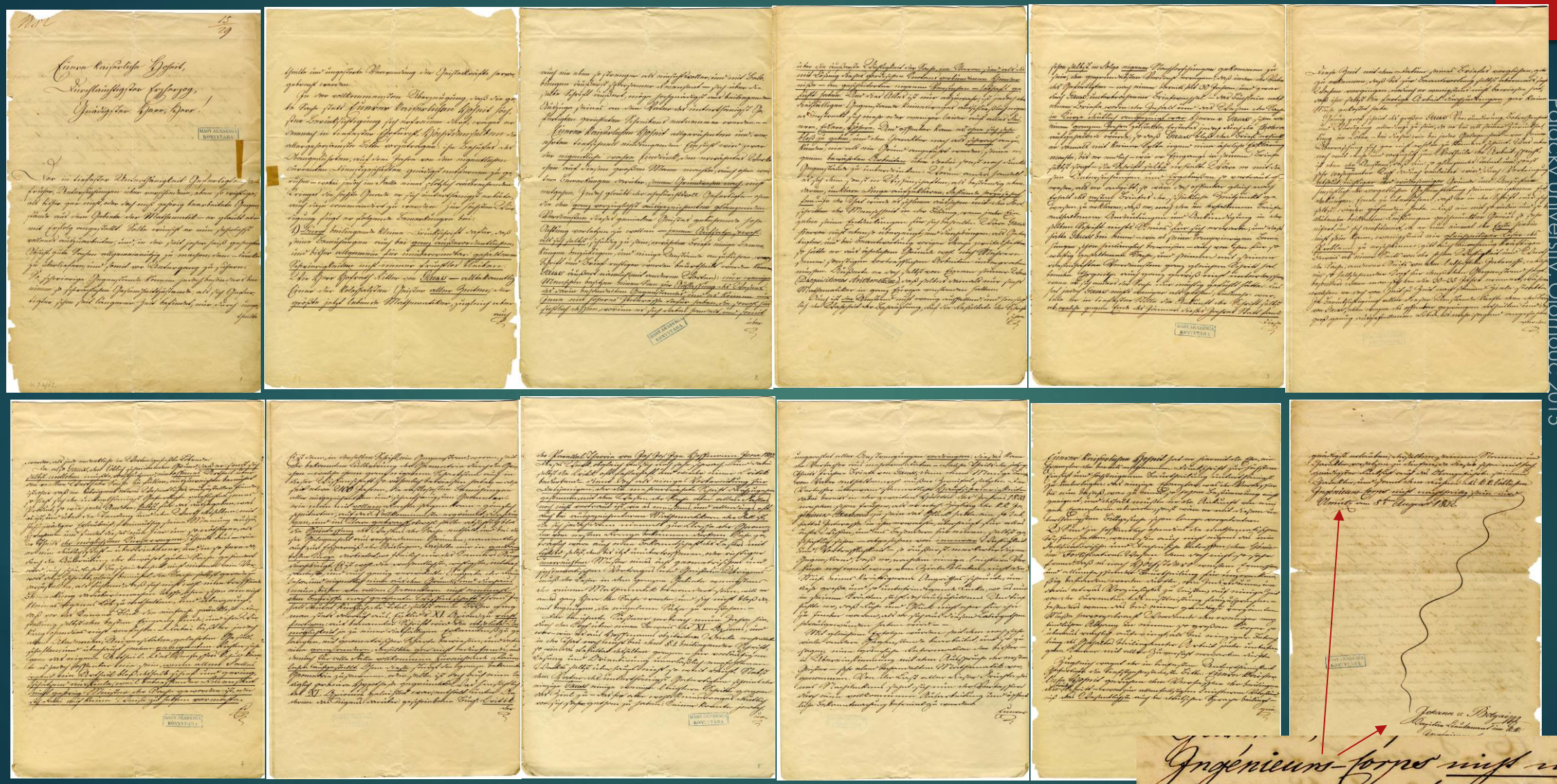
# 1832-1833, Olomouc

- **On March 14, 1832** he was promoted to captain.
- **From April 27, 1832 to June 15, 1833** he was the officer of engineering of the directorship of fortification of Olmütz (Olomouc).
- **On May 12, 1832** he left Lemberg for his next station of service, Olmütz.
- **On May 17, 1832** on the way around Bielitz his waggon fell in a ditch along the road, and he was taken taken senseless to a village near to Bielitz. His wounds and presumable commotion were treated for more than a month. He arrived to his post of service in Olmütz at the beginning of July 1832.
- **On June 16, 1833** the commission of supervision declared János Bolyai disabled and pensioned him off with yearly 400 forints.
- **In 1833** (or, according to a note of 1834 by János Bolyai, at the end of 1833) was published the second volume of the *Tentamen*of Farkas Bolyai. The first volume of this work, published in 1832 contained the *Appendix* by János.

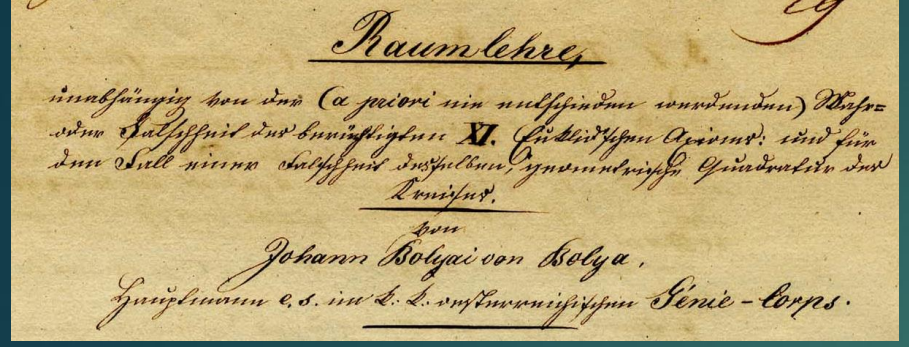


# A letter written by János Bolyai from Olomouc



Ingénieurs-Corps nicht unwichtig
Olmütz, am 8ten August 1832.

# Raumlehre-version, in Olomouc, 1832

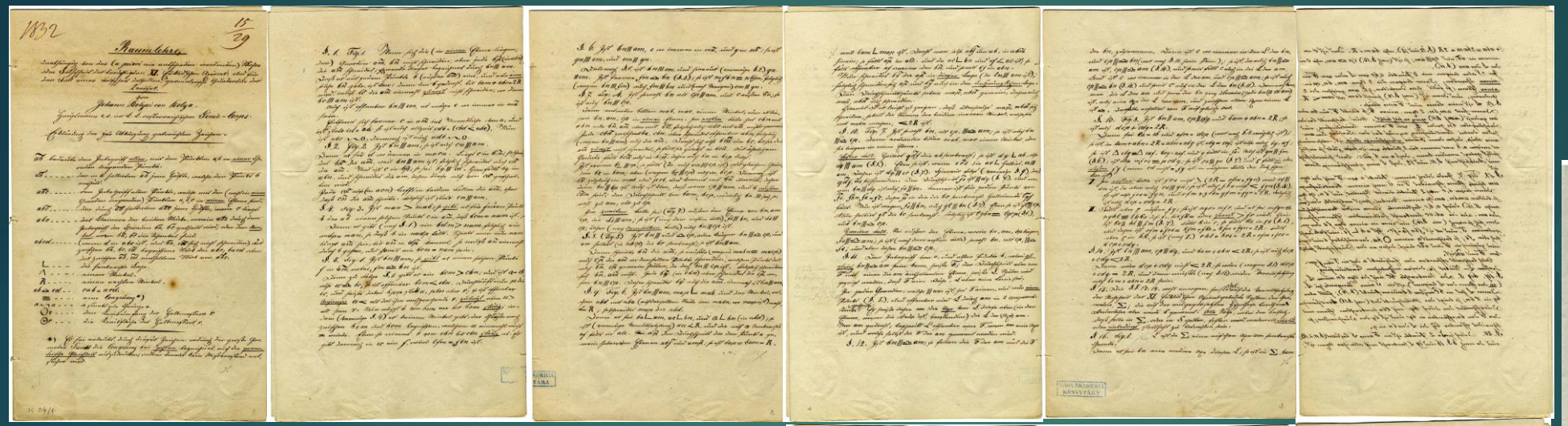

Raumlehre

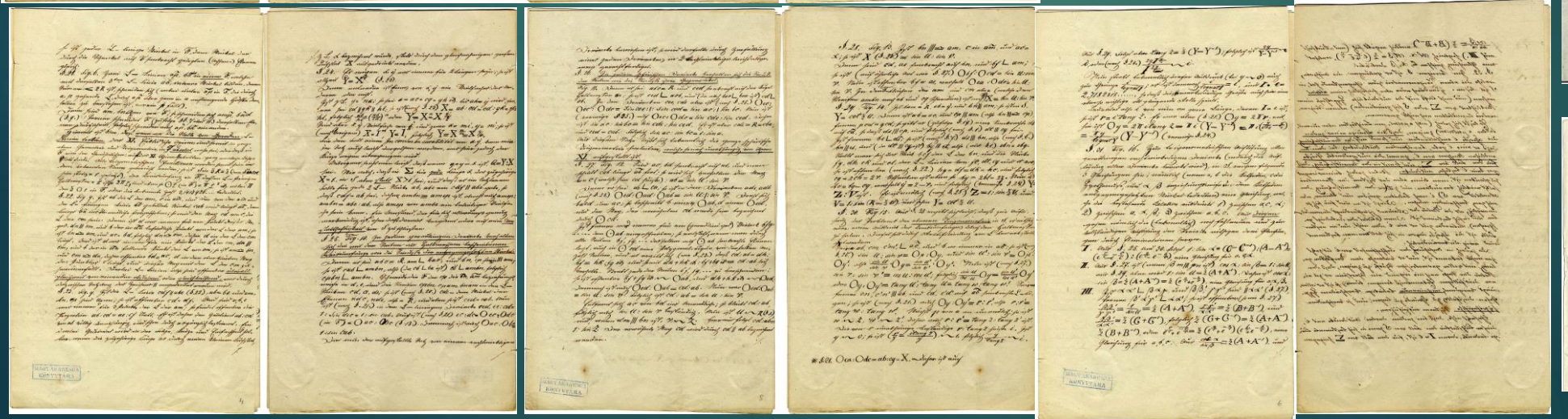
unabhängig von dem (a priori nie vollständig entschiedenen) Wahrheitsbegriffe des bekannten XI. Euklidischen Axioms; und für den Fall seiner Falschheit desselben, eigentümliche Grundlehre der Trigonometrie.

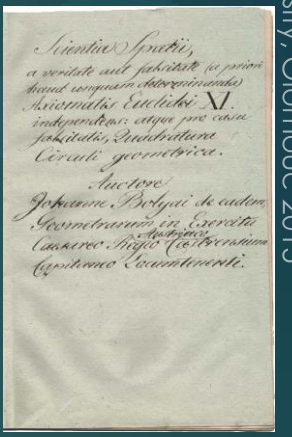
von

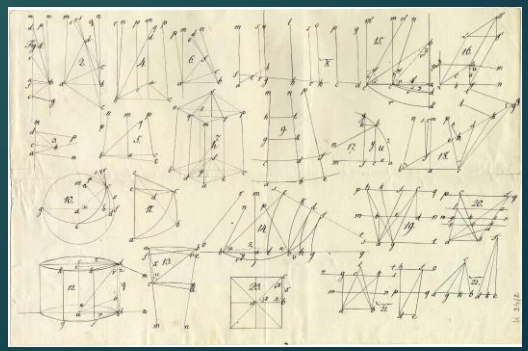
Johann Bolyai von Bolya,

Hauptmann e. h. im k. k. ingenieur-geographischen Genie-Corps.





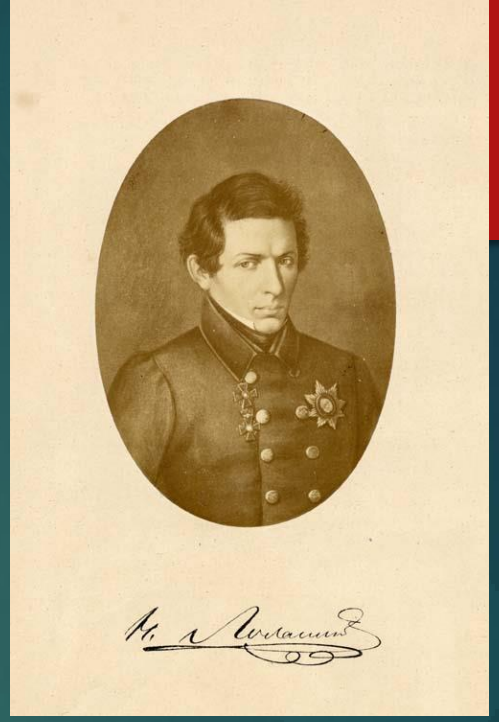

Scientiae Spatii,
a veritate aut falsitate (a priori haud unquam adornanda)
Axiomatis Euclidei XI.
independentiae atque pro casu falsitatis, Quadratura Circuli geometrica.
Auctore
Johanne Bolyai de eadem
Geometrarum in Exercitu Caesareo Regio Austriaco
Castrensium Comite Exercituum.



Palacký University, Olomouc 2015

# Bolyai and Lobachevski

- **On October 17, 1848** he received of his father the *Geometrische Untersuchungen zur Theorie der Parallellinien* by Nikolai Ivanovich Lobachevski. Later he wrote a critical study on it.
- *János studied Lobachevsky's work carefully and analysed it line by line, not to say word by word, with just as much care as he administered in working out the Appendix. The work stirred a real storm in his soul and he gave outlet to his tribulations in the comments added to the 'Geometrical Examinations'.*
- *The 'Comments' to the 'Geometrical Examinations' are more than a critical analysis of the work. They express the thoughts and anxieties of János provoked by the perusal of the book. They include his complaint that he was wronged, his suspicion that Lobachevsky did not exist at all, and that everything was the spiteful machinations of Gauss: it is the tragic lament of an ingenious geometrician who was aware of the significance of his discovery but failed to get support from the only person who could have appreciated his merits.*
- *In spite of his mental agitation amidst which János put observations to paper, he preserved enough objectivity to highly appreciate the work of his rival. In his comment to Theorem 35 he remarks that the proofs of Lobachevsky concerning spherical trigonometry bear the impress of genius and his work should be esteemed as a masterly achievement.*

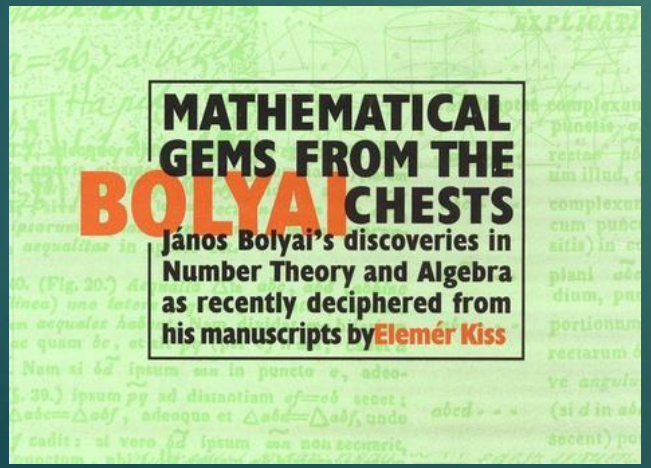


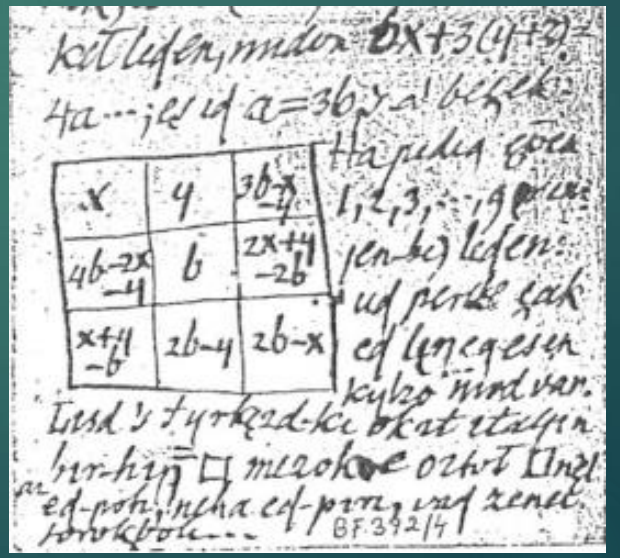
# Lobachevsky, student, and later on professor, and rector for 19 years of Kazan University

- One of the excellent professors who had been invited from Germany to Kazan, was Martin Bartels (1769 - 1833) who had been appointed as Professor of Mathematics. Bartels was a school teacher and friend of Gauss, and the two corresponded. A skilled teacher, Bartels soon interested Lobachevsky in mathematics. We do know that Bartels lectured on the history of mathematics and that he gave a course based on the text by Montucla. Since Euclid's *Elements* and his theory of parallel lines are discussed in detail in Montucla's book, it seems likely that Lobachevsky's interest in the Fifth Postulate was stimulated by these lectures. Lobachevsky attended this history course given by Bartels.
- Lobachevsky received a Master's Degree in physics and mathematics in 1811. In 1814 he was appointed to a lectureship and in 1816 he became an extraordinary professor. In 1822 he was appointed as a full professor.
- In 1827 Lobachevsky became rector of Kazan University, a post he was to hold for the next 19 years.
- His major work, *Geometriya* completed in 1823, was not published in its original form until 1909. On 11 February 1826, in the session of the Department of Physico-Mathematical Sciences at Kazan University, Lobachevsky requested that his work about a new geometry was heard and his paper *A concise outline of the foundations of geometry* was sent to referees.
- The text of this paper has not survived but the ideas were incorporated, perhaps in a modified form, in Lobachevsky's first publication on hyperbolic geometry.
- He published this work on non-euclidean geometry, the first account of the subject to appear in print, in 1829. It was published in the *Kazan Messenger* but rejected by Ostrogradski when it was submitted for publication by the St Petersburg Academy of Sciences.

# János Bolyai did further research in many areas of mathematics

- It was Elemér Kiss, Professor at Marosvásárhely, who refuted this opinion: Having consulted Bolyai's manuscripts left to us for about a decade he found significant mathematical 'gems' in them which could be considered new at their birth.
- Fermat's small theorem, the „Jeans' theorem", The Christmas theorem of Fermat, the Mersenne and Fermat primes, or Magic squares, only a few of the areas he has obtained original results.



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- János Bolyai, APPENDIX. Ed. F. Kárteszi, Akadémiai Kiadó, Budapest, 1987.
- Appendix. Scientiam Spatii. Bolyai János honlapja: [on line:http://bolyai.mtak.hu/hu/life.htm] seen 19 September 2015
- MacTutor History of mathematics Archive: [on line: http://www-groups.dcs.st-and.ac.uk/~history/index.html]
- Kiss Elemér: *Mathematical Gems from the Bolyai Chests, Akadémiai Kiadó, Budapest, Typotex Kiadó, Szeged, 1999.*

# Thank You for your kind attention!